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Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let $U = ax^4 + by^4 + cz^4 + 4(a_2x^3y + a_3x^3z + b'_3y^3z + b_1y^3x + c_1z^3x + z_3z^3y)$

$$+6(dy^2z^2 + ez^2x^2 + fx^2y^2) + 12xyz(lx + my + nz)$$

be the ternary quartic. The required invariant written out in full is

$$\frac{d^4u}{dx^4}\cdot\frac{d^4u}{dy^4}\cdot\frac{d^4u}{dz^4} - 4\Big(\frac{d^4u}{dx^4}\cdot\frac{d^4u}{dy^3\,dz}\cdot\frac{d^4u}{dz^3\,dy} + \frac{d^4u}{dy^4}\cdot\frac{d^4u}{dz^3\,dx}\cdot\frac{d^4u}{dx^3\,dz} + \frac{d^4u}{dz^4}\cdot\frac{d^4u}{dx^3\,dy}\cdot\frac{d^4u}{dy^3dx}\Big)$$

$$+ 3\Big[\frac{d^4u}{dx^4}\Big(\frac{d^4u}{dy^2\,dz^2}\Big)^2 + \frac{d^4u}{dy^4}\Big(\frac{d^4u}{dx^2\,dz^2}\Big)^2 + \frac{d^4u}{dz^4}\Big(\frac{du^4}{dx^2\,dy^2}\Big)\Big]$$

$$+ 4\Big(\frac{d^4u}{dx^3dy}\cdot\frac{d^4u}{dy^3\,dz}\cdot\frac{d^4u}{dz^3\,dx} + \frac{d^4u}{dx^3\,dz}\cdot\frac{d^4u}{dy^3dx}\cdot\frac{d^4u}{dz^3\,dy}\Big) - 12\Big(\frac{d^4u}{dx^3dy}\cdot\frac{d^4u}{dxdydz^2}\cdot\frac{d^4u}{dy^2\,dz^2}$$

$$+ \frac{d^4u}{dx^3\,dz}\cdot\frac{d^4u}{dx\,dy^2dz}\cdot\frac{d^4u}{dy^2\,dz^2} + \frac{d^4u}{dy^3dx}\cdot\frac{d^4u}{dx\,dy\,dz^2}\cdot\frac{d^4u}{dz^2\,dx^2} + \frac{d^4u}{dy^3\,dz}\cdot\frac{d^4u}{dx^2\,dy\,dz}\cdot\frac{d^4u}{dz^2\,dx^2}$$

$$+ \frac{d^4u}{dz^3\,dx}\cdot\frac{d^4u}{dx\,dy^2\,dz}\cdot\frac{d^4u}{dx^2\,dy^2} + \frac{d^4u}{dz^3\,dy}\cdot\frac{d^4u}{dx^2\,dy\,dz}\cdot\frac{d^4u}{dx^2\,dy^2}\Big) + 12\Big(\frac{d^4u}{dx^2dydz}\cdot\frac{d^4u}{dy^3dx}\cdot\frac{d^4u}{dz^3\,dx}$$

$$+ \frac{d^4u}{dx\,dy^2dz}\cdot\frac{d^4u}{dz^3\,dy}\cdot\frac{d^4u}{dx^3dy} + \frac{d^4u}{dx\,dy\,dz^2}\cdot\frac{d^4u}{dx^3\,dz}\cdot\frac{d^4u}{dy^3\,dz}\Big) + 12\Big[\frac{d^4u}{dy^2\,dz^2}\cdot\Big(\frac{d^4u}{dx^2\,dy\,dz}\Big)^2$$

$$+ \frac{d^4u}{dz^2\,dx^2}\cdot\Big(\frac{d^4u}{dx\,dy^2\,dz}\Big)^2 + \frac{d^4u}{dx^2\,dy^2}\cdot\Big(\frac{d^4u}{dx\,dy\,dz^2}\Big)^2\Big] + 6\frac{d^4u}{dy^2\,dz^2}\cdot\frac{d^4u}{dy^2\,dx^2}\cdot\frac{d^4u}{dx^2\,dz^2}$$

$$- 12\,\frac{d^4u}{dx^2\,dy\,dz}\cdot\frac{d^4u}{dx\,dy^2\,dz}\cdot\frac{d^4u}{dx\,dy\,dz^2}.$$

$\therefore abc - 4(ab_3c_2 + bc_1a_3 + ca_2b_1) + 3(ad^2 + be^2 + cf^2) + 4(a_2b_3c_1 + a_3b_1c_2)$ $-12(a_2nd + a_3md + b_1ne + b_3le + c_1mf + c_2lf) + 12(lb_1c_1 + mc_2a_2 + na_3b_3) + 12(dl^2$ $+ em^2 + fn^2) + 6def - 12lmn.$

139. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

Given the roots of a binary cubic, to find the roots of its two inbependent covariants.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Let $ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$ be the cubic. The two independent covariants are,

$$(ac-b^2)x^2+(ad-bc)xy+(bd-c^2)y^2=0. \qquad (1)$$

$$(ad^2-3bcd+2c^3)x^3+3(2b^2d-acd-bc^2)x^2y+3(2ac^2-abd-b^2c)xy^2$$
$$+(a^2d-3abc+2b^3)y^3=0. \qquad (2)$$

Let $x/y=z$, then $az^3+3bz^2+3cz+d=0$ is the cubic. Let α, β, γ be the gamma roots of this cubic. Then

$$A=(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2=2(\alpha^2+\beta^2+\gamma^2)-2(\alpha\beta+\alpha\gamma+\beta\gamma)$$
$$=-(18/a^2)(ac-b^2);$$

$$B=2[\alpha(\beta-\gamma)^2+\beta(\gamma-\alpha)^2+\gamma(\alpha-\beta)^2]=-(18/a^2)(ad-bc);$$

$$C=\alpha^2(\beta-\gamma)^2+\beta^2(\gamma-\alpha)^2+\gamma^2(\alpha-\beta)^2=-(18/a^2)(bd-c^2).$$

$\therefore$ The quadratic covariant is $Az^2+Bz+C=0$. The roots of this equation are $[B\pm\sqrt{(B^2-4AC)}]/2A$. Substituting the values of A, B, C in terms of α, β, γ, the roots are

$$\frac{\alpha(\beta-\gamma)^2+\beta(\gamma-\alpha)^2+\gamma(\alpha-\beta)^2\pm(\alpha-\beta)(\gamma-\alpha)(\beta-\gamma)\sqrt{(-3)}}{(\alpha-\beta)^2+(\gamma-\alpha)^2+(\beta-\gamma)^2}.$$

$$D=2\beta\gamma-\alpha\beta-\alpha\gamma=3\beta\gamma-(\beta\gamma+\alpha\beta+\alpha\gamma)=3\beta\gamma-3c/a;$$
$$E=2\gamma\alpha-\beta\gamma-\beta\alpha=3\gamma\alpha-3c/a;$$
$$F=2\alpha\beta-\gamma\alpha-\gamma\beta=3\alpha\beta-3c/a;$$
$$G=\alpha+\beta-2\gamma=\alpha+\beta+\gamma-3\gamma=-3b/a-3\gamma;$$
$$H=\beta+\gamma-2\alpha=-3b/a-3\alpha;$$
$$K=\gamma+\alpha-2\beta=-3b/a-3\beta.$$

$$\therefore DEF=27(\alpha\beta-c/a)(\gamma\alpha-c/a)(\beta\gamma-c/a)=(27/a^3)(ad^2-3bcd+2c^3);$$
$$GHK=-27(\alpha+b/a)(\beta+b/a)(\gamma+b/a)=(27/a^3)(a^2d-3abc+2b^3);$$
$$DEG+DFK+EFG=-27[(\beta\gamma-c/a)(\gamma\alpha-c/a)(\gamma+b/a)+(\beta\gamma-c/a)$$
$$\times(\alpha\beta-c/a)(\beta+b/a)+(\gamma\alpha-c/a)(\alpha\beta-c/a)(\alpha+b/a)]$$
$$=-(81/a^3)(2b^2d-acd-bc^2);$$

$$DGK+EGH+FHz=27[(\beta\gamma-c/a)(\gamma+b/a)(\beta+b/a)+(\gamma\alpha-c/a)(\alpha+b/a)$$
$$\times(\gamma+b/a)+(\alpha\beta-c/a)(\alpha+b/a)(\beta+b/a)]$$
$$=(81/a^3)(2ac^2-abd-b^2c).$$

$\therefore$ The cubic covariant is

$$DEFz^3+(DEG+DFK+EFH)z^2+(DGK+EGH+FHK)z+GHK=0.$$

$$\therefore z^3+(G/F+K/E+H/D)z^2+(GK/EF+GH/DF+HK)DE)z+GHK/DEF=0.$$

$$\therefore (z+G/F)(z+K/E)(z+H/D)=0.$$

$\therefore$The roots are $-G/F$, $-K/E$, $-H/D$, or

$$\frac{2\gamma-\alpha-\beta}{2\alpha\beta-\gamma\alpha-\gamma\beta},\quad \frac{2\beta-\alpha-\gamma}{2\gamma\alpha-\beta\gamma-\beta\alpha},\quad \frac{2\alpha-\beta-\gamma}{2\beta\gamma-\alpha\beta-\alpha\gamma}.$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

185. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Without introducing radicals, eliminate x and y from the equations

$$ax^2+bx+c=0,\ ay^2+by+d=0,\ ax^2y^2+bxy+e=0.$$

186. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Eliminate x and y from the equations

$$ax^3+bx^2+cx+d=0,$$
$$ay^3+by^2+cy+e=0,$$
$$ax^3y^3+bx^2y^2+cxy+f=0,$$

the eliminant to be rational in d, e, f.

GEOMETRY.

207. Proposed by W. W. HART, University High School, Chicago, Ill.

According to Gauss the circumference of a circle can be divided into n equal parts by ruler and compass when and only when n is a prime of the form $2^{2^p}+1$.

The following construction gives good partial results for n equals *any* integer. If AB is the diameter of the circle, and C is the vertex of the equilateral triangle ABC, and if D is a point on AB at the distance $2AB/n$ from A, then draw the line CD cutting the circle at E and F; E being the more remote from C. $AE=1/n$ circumference approximately. For low values of n this method is very practical; is it practical in general? How great is the error?

208. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Tangents drawn to two confocal parabolae from a point on the common tangent intersect at the same angle as the axes of the parabolae.